

# Turbulence lifetimes: what can we learn from the physics of glasses?

Olivier Dauchot<sup>1</sup> and Eric Bertin<sup>2</sup>

<sup>1</sup> *EC2M, ESPCI-ParisTech, UMR Gulliver 7083 CNRS, 75005 Paris, France*

<sup>2</sup> *Université de Lyon, Laboratoire de Physique, ENS Lyon, CNRS, 46 Allée d'Italie, 69007 Lyon, France*

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In this note, we critically discuss the issue of the possible finiteness of the turbulence lifetime in subcritical transition to turbulence in shear flows, which attracted a lot of interest recently. We briefly review recent experimental and numerical results, as well as theoretical proposals, and compare the difficulties arising in assessing this issue in subcritical shear flow with that encountered in the study of the glass transition. In order to go beyond the purely methodological similarities, we further elaborate on this analogy and propose a qualitative mapping between these two apparently unrelated situations, which could possibly foster new directions of research in subcritical shear flows.

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## I. INTRODUCTION

Statistical physics has devoted a lot of effort to the study of fully developed turbulence, but much less to that of the transition to turbulence [1], which occurs when the Reynolds number, the ratio of the advection time to the viscous time, is increased. The transition is commonly observed in flow regimes lacking linear instability and is referred to as globally subcritical [2–4].

The plane Couette flow, driven by two plates moving parallel to each other in opposite directions is linearly stable at all Reynolds number and, as such, is the epitome of globally subcritical transitions [5]. Other flows usually transit to turbulence before linear instability sets in. These include the circular Poiseuille flow (cPf) and the plane Poiseuille flow (pPf), which are driven by a pressure gradient respectively along a circular pipe or between two parallel plates, as well as the counter-rotating Taylor Couette flow (TCf), driven by two concentric cylinders rotating in opposite directions. In all these cases, the transition is particularly delicate to understand owing to its abrupt character. Many non-trivial unstable solutions of the Navier-Stokes equations coexist in phase space with the laminar flow, leading to a complex spatio-temporal dynamics in real space including the nucleation and the growth or decay of turbulent domains called 'puffs' (pPf) or 'spots' (pCf) –see, e.g., [6, 7] for cPf, [8–10] for pCf, [11] for pPf, and [12, 13] for TCf.

A recent surge of interest has been motivated by the audacious proposal that shear flow turbulence could remain transient up to arbitrarily large Reynolds number, opening ways towards a better control of such turbulent regimes [14]. This proposal was motivated by new experimental and numerical observations in cPf [14] regarding the statistics of turbulent lifetimes, in contradiction with those previously obtained in cPf [15, 16] and pCf [17, 18]. This contradiction, together with the experimental discovery of a spectacular long wavelength periodic organization of the laminar-turbulent coexistence in pCf and TCf [19, 20], has motivated new experiments in TCf [21] and cPf [22, 23], the development of various models [24–28] and an impressive number of numerical

studies [22, 29–40]. As a result, a better knowledge of the organization of the phase space and some comprehension of the mechanisms at play in the coexistence dynamics has been gained. However, whether the turbulence lifetime remains finite or diverges at some critical Reynolds number is still a matter of debate.

Interestingly, the possible existence of finite, yet extremely large relaxation times, together with the coexistence in phase space of many unstable solutions (saddles from a geometric point of view), is reminiscent of the physics of glasses (see, e.g., [41–43]). Whether the structural relaxation times of a glass really diverges at a given finite temperature, or remains very large but finite at any positive temperature, is an important issue related to the existence of a genuine phase transition to an ideal glass state. After several decades of intense activity, a majority of the community has come to the general agreement that no fitting procedure, however cautious, could bring a definitive answer to that question [44]. In spite of this apparent failure, new (and perhaps even more interesting) questions have appeared as a result of this research effort, and have driven the field of glasses towards important conceptual progresses [43].

Here, in the legacy of Y. Pomeau [45], we would like to draw again the attention of the statistical physics community towards the old standing problem of the transition to turbulence. On the one hand, many new concepts have been developed in the physics of glasses (and more generally of disordered systems) in the last twenty-five years, since the publication of this seminal paper. On the other hand, the development of experimental techniques such as Particle Image Velocimetry, and the exponential increase of the numerical capacities now give access to never achieved before datasets about the transition to turbulence. In such circumstances, one may hope significant progress to emerge again from the confrontation of both fields.

As a first modest contribution in this direction, we wish to discuss, in the light of a simple model inspired by the so-called Random Energy Model [46, 47] introduced to describe disordered systems, what scenarios could lead to diverging turbulence lifetimes at some finite Reynolds

number. Our aim here is to illustrate the possible richness of the analogy, and to possibly inspire new experimental protocols and new data analysis methods. It might also point at deeper connections between these two active fields of nonlinear and statistical physics. The paper is organized as follows. We shall first propose a concise review of the major results dealing with subcritical transition to turbulence. We then introduce our model, before discussing the possible outcomes of the analogy.

## II. FINITE VERSUS DIVERGING LIFETIMES

We first briefly review the experiments and direct numerical simulations reporting the increase of the turbulence lifetime when the Reynolds number is increased. We then describe two complementary numerical approaches, a low dimensional (mean field-like) one, focusing on the phase space structure, and a spatially extended one focusing on the real space dynamics.

### A. Experimental and Numerical observations

To our knowledge, the first systematic measurement of turbulence lifetimes was conducted in the pCf [17, 18]. Two different kinds of experiments have been performed, differing by the way the initial condition is prepared. In what we shall call type-A experiments, the Reynolds number,  $R$ , is set to its value of interest and the laminar flow is disturbed locally at the initial time. In type-B experiments, a turbulent flow is prepared at high  $R$ , and quenched at the initial time down to the  $R$  value of interest. In both cases, one monitors the evolution of the turbulent fraction  $f_T(t)$ , which characterizes the coexistence dynamics of laminar and turbulent domains (see Fig 1). For  $R > R_g$ ,  $f_T(t)$  fluctuates around some average value, which remains finite on experimental timescales. For  $R < R_u$ ,  $f_T(t)$  relaxes towards zero, without displaying any transient regime. In between, for  $R_u < R < R_g$ ,  $f_T(t)$  exhibits a first rapid decay, followed by a long transient quasi-steady regime, before a large fluctuation sets it to zero. The lifetime of these transients are exponentially distributed and the average value  $\tau$  was reported to diverge like  $(R_g - R)^{-1}$ .

The cPf was later investigated in various ways. In [6, 16] a puff was generated inside a constant flow rate pipe flow by introducing a short duration perturbation. Then  $R$  was reduced and the subsequent evolution of the puff was monitored as it progressed downstream. The probability of observing a localized disturbed region of flow as a function of distance downstream is exponential and the time required for half of the initial states to decay,  $\tau_{1/2}$ , was reported to diverge like  $(R_c - R)^{-1}$ , in agreement with the observations made in the pCf. Other protocols lead to the same conclusions [6, 16].

However, these results were challenged later by another experimental study [14]. In a pressure driven flow

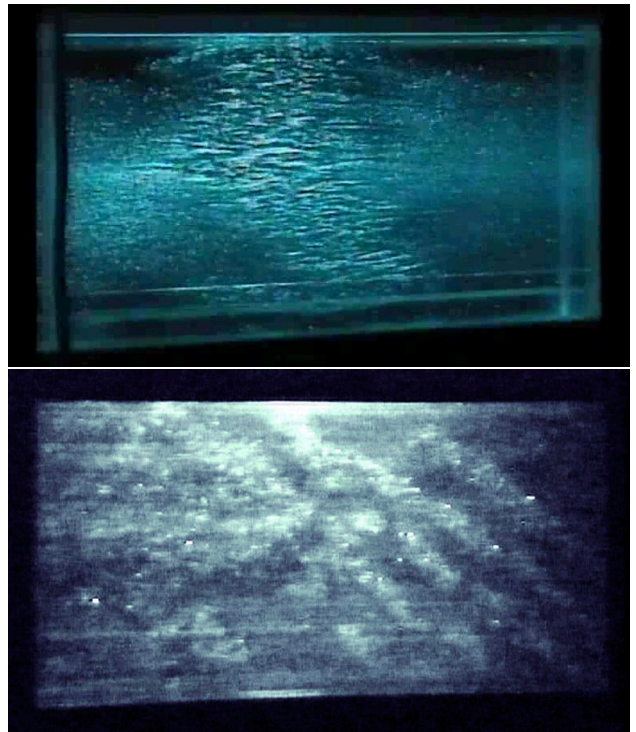


FIG. 1: Typical snapshots of the laminar-turbulence coexistence at intermediate Reynolds number in plane Couette flow. Top: turbulent spot in a small aspect ratio setup, created by a localized perturbation. Bottom: Large scale coexistence in a large aspect ratio setup, following a quench from high Reynolds number.

through a very long pipe, the authors could record much longer dimensionless observation times. They could determine the probability to be turbulent after a time period given by the distance between the perturbation location and the outlet, as a function of flow rate. For short times, the data are within the error bars of [6, 16] but for longer times they deviate from the divergent behavior reported above and are better represented by an exponential variation:  $\tau = \exp(aR + b)$ , without singularity (here and in what follows,  $a$  and  $b$  denote generic fit parameters). Finally in a recent experimental study of turbulence in pipe flow spanning height orders of magnitude in time, drastically extending all previous investigations, it was claimed that the turbulent state remains transient, with a mean lifetime, which depends super-exponentially with the Reynolds number:  $\tau \propto \exp(\exp(aR + b))$  [48].

Intense numerical simulations of the cPf have also been conducted, but did not clarify the situation. In [15, 30] a diverging behavior of the turbulent lifetimes compatible with the experimental results of [6, 16] is reported. Later in [31], the authors –one of which is common to [15]– conducted new simulations and reanalyzed older data, concluding to an exponential dependence such as the one reported in [14]. Altogether despite intense experimental and numerical effort, no definitive answer regarding the divergence or finiteness of turbulence lifetime could be obtained from the fit of data by phenomenological func-

tional forms.

As stated in the introduction, this issue is not specific to the transition to turbulence. When a liquid is suddenly quenched below its crystallization temperature and if crystallization can be avoided, the liquid enters a state, called supercooled liquid, in which the relaxation time increases by several orders of magnitude over a limited range of temperature [41]. A divergence at a finite temperature of the relaxation time would signal an ideal glass transition, and would thus be of high interest, at least at a conceptual level. Despite huge efforts made to measure the variations of the relaxation time over an experimental window of more than ten decades, no clear consensus has been obtained yet. More precisely, the available data are both consistent with fits including a divergence at a finite temperature  $T_c > 0$ , and with fits diverging only at  $T = 0$  [44].

The same difficulty is also expected to appear in the context of turbulence. We illustrate this point on experimental data recently obtained in the case of the TCf [21], when only the external cylinder is rotating. The TCf is then, like the pCf, linearly stable for all  $R$ . Also, because the TCf is a closed flow, one can record very long times. In this experiment, the angular rotation of the external cylinders fixes the Reynolds number. The flow was perturbed by rapidly accelerating the inner cylinder in the direction opposite to the rotation of the outer cylinder and immediately stopping it. After a short regime of featureless turbulence, the flow exhibits long transients characterized by the coexistence of laminar and turbulent domains, before eventually relaxing towards the laminar flow. The distribution of lifetimes is again exponential, and the authors argue that the mean turbulent lifetime does not diverge and rather behaves in the transitional regime as a double exponential  $\tau \propto \exp(\exp(aR + b))$ , as observed in the cPf [48].

It is interesting to note that in the oldest experiments, the debate about the functional dependence of the average turbulent lifetime on the Reynolds number was concentrating on the choice between the two following forms:

$$\tau/\tau_0 = \exp(R/R_0) \quad (1)$$

$$\tau/\tau_0 = \left( \frac{R_c}{R_c - R} \right)^\alpha, \quad \alpha > 0, \quad (2)$$

whereas the most recent experiments, both in the case of the cPf [48] and the TCf [21], have access to much longer experimental timescales and point at a double exponential behavior. This last functional form ensures a very fast increase of  $\tau$  without singularity, and could give the impression that it solves the above debate. However as learnt from the physics of glasses, the debate has actually been shifted towards two alternative functional forms, namely:

$$\ln(\tau/\tau_0) = \lambda \exp(R/R_0) \quad (3)$$

$$\ln(\tau/\tau_0) = \lambda \left( \frac{R_c}{R_c - R} \right)^\alpha, \quad \alpha > 0. \quad (4)$$

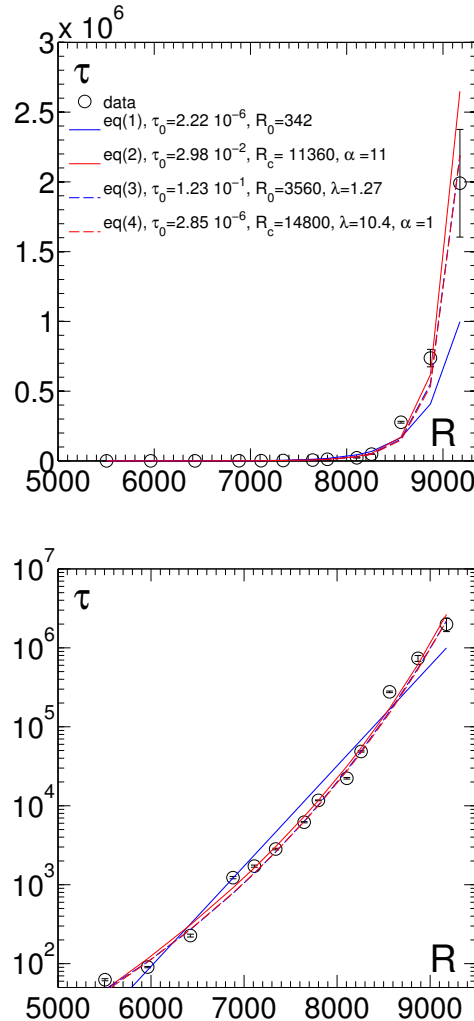


FIG. 2: Probing finite lifetime experimentally (color online): Relaxation lifetimes of turbulent initial conditions in a Taylor-Couette flow, with rotating external cylinder and internal cylinder at rest (data from [21]). Four possible fits are proposed as indicated in legend. Top and bottom panels are in lin-lin and lin-log scales respectively. Times are given in units of  $d/r_0\omega_0$ , where  $d$  is the gap between the two cylinders,  $r_0$  is the radius of the external cylinder and  $\omega_0$  its angular velocity. All fits were performed using a standard least square fit procedure. Fit with Eq. (1) was obtained imposing a linear fit of  $\log(\tau)$  vs.  $R$ . Fit with Eq. (2) was obtained imposing a linear fit of  $\log(\tau)$  vs.  $\log(R_c/(R_c - R))$ , optimizing the fit quality on  $R_c$ . Fit with Eq. (3) was obtained imposing a linear fit on  $\log(\tau)$  vs.  $\exp(R/R_0)$ , optimizing the fit quality on  $R_0$ . Finally, fit with Eq. (4) was obtained imposing a linear fit of  $\log(\tau)$  vs.  $R_c/(R_c - R)$ , optimizing the fit quality on  $R_c$ . In this last case,  $\alpha$  was thus set to one. It was checked that other values of  $\alpha$  up to 3 cannot be discriminated. For Eq. (1), the regression coefficient  $r_2$  is equal to 0.971. For all the other cases,  $r_2 = 0.994$ .

As a matter of fact, Eq. (4), which has (to the best of our knowledge) not yet been proposed in the context of the transition to turbulence, is a very standard form called the Vogel-Fulcher-Tammann (VFT) law in the physics of glasses [44].

Figure 2 displays the data obtained in TCf [21] –which are available online as supplementary material– together with possible fits by the four functional forms proposed above. Note that we have only performed global fits of the data, without trying to extract various regimes and crossovers as can be done in the case of glasses [44]. One clearly observes that indeed the relevant variable to describe the growth of the turbulent lifetimes is  $\ln(\tau)$ , as soon as really large times are considered. However, one also sees that apart from the simple exponential form Eq. 1, all other descriptions are not discriminable, so that there is no definitive way to rule out or confirm the existence of a singularity. One faces the same difficulty as in the physics of glasses: the lifetimes to be measured become very large, which makes it difficult to accumulate significant statistics. The experimental results are thereby confined in a finite range of Reynolds number or temperature, from which even with high-quality data sets, the divergence of a characteristic time cannot convincingly be determined from fits.

Before concluding this section, let us mention that the double-exponential form Eq. 3 has been justified on the basis of extreme value statistics [49]. However, as stated by the authors, the argument is only local, as it involves an expansion in  $R$  around a given reference value. Hence no clear conclusion can be drawn from the theoretical argument of [49] on the issue of the divergence of  $\tau$  at a finite or infinite value of  $R$ . Finally, let us emphasize that for now, we have left aside all issues related to finite size effects, which in turn can severely alter the functional dependence of time and length scales in transitional regimes.

## B. Low dimensional vs. spatially extended models

The globally subcritical transition to turbulence is by definition controlled by solutions of the Navier-Stokes equation, which do not branch continuously from the laminar flow solution when the Reynolds number is increased. These solutions –of various kinds, stationary states, traveling waves, or more complex coherent structures– are unstable and form hyperbolic states, with stable and unstable manifolds. Early indication of the existence of these solutions were reported in pCf, both numerically [50–52] and experimentally [53]. The intricate network made of these manifolds and their connections then serves as a skeleton for the turbulent flow.

In principle one would like to collect all such states, estimate their dynamical weight and calculate statistical averages from periodic orbit theory [54]. In practice, one must restrict the analysis to low-dimensional models [55–59] or to simulations [14, 15, 24, 31, 59, 60], performed in the so-called minimal flow unit assumption [61]. Doing so, it was shown that the regions of initial conditions for which long lifetimes exhibit strong fluctuations and a sensitive dependence on initial conditions were separated from the regions with short lifetimes and smooth varia-

tions by a border, the so-called “edge of chaos”. Later, some exact solutions with codimension-1 stable manifolds have been identified as edge states, that is, solutions that locally form the stability boundary between laminar and turbulent dynamics [34–36]. These important results contributed to make concrete the picture borrowed from dynamical system theory of a turbulent repeller, separated from the laminar state by a set of edge states connected through heteroclinic manifolds. In particular the existence of the above non trivial solutions has served to understand the exponential distribution of lifetimes in the transitional regime.

Unfortunately the above picture does not bring a definitive answer to the issue of the existence of a finite critical Reynolds number above which turbulence is sustained. As argued in [25, 27, 62], the reason is that the dynamics being either projected on a small set of modes or limited to small computational domains with periodic boundary conditions, it cannot capture the genuinely spatiotemporal coexistence of laminar and turbulent states observed in open and unbounded flows. In particular, it can neither capture the long wavelength modulation of turbulent intensity, nor the regime of alternating laminar and turbulent stripes, first observed experimentally in pCf and TCf [19, 20] then reproduced numerically in pCf [29, 33, 36, 38, 40, 63].

As a matter of fact it is known for long that, according to the scenario called spatiotemporal intermittency (STI [64]), transient chaotic local states of a distributed system may evolve into a sustained turbulent global state due to spatial couplings [65–68]. Following this path, it was demonstrated that a simple 1D-model of cPf, composed of coupled maps, does indeed captures remarkably well the character of the turbulent pipe flow in the transitional regime and exhibits a critical transition towards sustained turbulence via spatiotemporal intermittency [28]. The transition is further believed to belong to the Directed Percolation class [65, 66, 69], as already suggested in [45] for pCf, and recently reconsidered in cPf [70].

Finally, it was shown by means of fully resolved direct numerical simulations of the Navier Stokes equation, that there exists a crossover length scale of the order of  $10^2$  times the cross-stream length below which the spatiotemporal processes at play in large-scale simulations and experiments are not captured [27]. Since then, a number of numerical investigations of large aspect ratios cPf and pCf have reproduced the complex spatio-temporal coexistence of laminar and turbulent states, and identified the first hydrodynamics mechanisms at play [22, 36, 37, 40].

## C. Summary

Let us now recast the essential features of subcritical transition to turbulence, which will feed the analogy with the glass transition that we would like to propose and illustrate on a simple model:

- *Subcriticality*: While the laminar flow is stable against infinitesimal perturbations, finite amplitude perturbations may trigger an abrupt transition towards a disordered flow. Such disordered flow can also be obtained by quenching fully turbulent flows.
- *Spatio-temporal intermittency*: This disordered flow is made of turbulent domains, which move, grow, decay, split and merge leading to spatio-temporal intermittency, that is a coexistence dynamics in which active/turbulent regions may invade absorbing/laminar ones, where turbulence cannot emerge spontaneously.
- *Transients and Meta-stability*: For large enough Reynolds number this disordered flow has long lifetimes, which are distributed exponentially. Whether the associated characteristic time diverges at a finite Reynolds number is still a matter of debate. For low Reynolds number, say  $R < R_u$ , or small enough disturbances, the flow relaxes rapidly towards the laminar flow.
- *Unstable states*: When increasing the Reynolds number a larger and larger number of unstable finite amplitude solutions appear in phase space. Some have been identified as edge states separating the others from the laminar state.

On the basis of these observations, we now propose an analogy with the glass transition, the goal of which is twofold. It offers a new framework to discuss possible scenarios which could lead to diverging turbulence lifetimes at some finite Reynolds. In addition, it may also inspire new analysis of the subcritical transition to turbulence, and might drive this field of research towards important, if not conceptual, methodological progresses.

### III. A SIMPLE MODEL FOR THE DIVERGENCE OF LIFETIMES

A first analogy with the study of glasses has been emphasized above at the methodological level, when studying the divergence of turbulence lifetime from empirical data. In this section, we further develop the analogy, and show that it can also be fruitful to some extent at the conceptual level, within the complex landscape picture. For illustration purposes, we introduce a simple model that captures, as an essential ingredient, the wandering on a complex landscape, and yields interesting predictions for the dependence of the turbulent lifetime on the Reynolds number.

#### A. The landscape picture

One of the main interests of the analogy with glasses is related to the energy landscape picture [71]. In glasses,

the transition between the liquid state and the crystal occurs, when lowering the temperature, through the emergence of a slowly relaxing glassy state (the super-cooled liquid) [41], resulting from the wandering of the phase-space point representing the system in a complex energy landscape [72], mostly composed of many saddles [42, 73, 74] (though local minima also play an important role at low enough temperature). The most striking feature of the glass transition, the rapid increase of the relaxation time, by several orders of magnitude over a moderate range of temperature is interpreted as a consequence of this complex dynamics in phase space.

Given the experimental and numerical evidences that we have reviewed in the previous section, it is tempting to see in the complex phase-space of the spatially extended dynamical systems an analog of the complex energy landscape of glasses. Though numerical investigations of turbulent flows have not been able yet to characterize the number of unstable solutions as a function of the volume of the flow, the analogy with glasses suggests that this number of solutions may grow exponentially with the volume of the system. Characterizing the state of the flow by its turbulent energy per unit volume,  $\varepsilon = E/V$  (that is the excess kinetic energy with respect to the laminar flow), we assume that the number  $\Omega_V(\varepsilon, R)$  of unstable solutions at a given energy density  $\varepsilon$  and Reynolds number  $R$  grows exponentially with the volume  $V$  according to

$$\Omega_V(\varepsilon, R) \sim e^{Vs(\varepsilon, R)}, \quad (5)$$

thus defining an entropy density  $s(\varepsilon, R)$ . At low Reynolds number, no unstable states exist, so that we assume that the entropy  $s(\varepsilon, R)$  is equal to zero for all  $\varepsilon > 0$  if  $R$  is less than a characteristic value  $R_u$ . For  $R > R_u$ , we assume that  $s(\varepsilon, R) > 0$  on an interval  $\varepsilon_{\min}(R) < \varepsilon < \varepsilon_{\max}(R)$ , and  $s(\varepsilon, R) = 0$  otherwise, meaning that unstable states exist only in the energy interval  $(\varepsilon_{\min}, \varepsilon_{\max})$ .

#### B. Diffusion in the energy landscape: the REM analogy

Turning to dynamics, we assume on the basis of the experimental and numerical observations reported in section IIB that the turbulent flow spends most of its time close to unstable solutions, and that the evolution of the flow can be considered as a succession of jumps between different unstable solutions. If however the flow ends up in the laminar state, the evolution stops until an external perturbation is imposed. Taking into account the presence of the absorbing laminar state is obviously essential to determine the lifetime of the turbulent flow. This will be the focus of Sect. III C. Yet, in a first stage, it is interesting to consider the evolution of the turbulent flow in the absence of the absorbing laminar state, in order to make the analogy with glass models emerge more clearly.

As it is unlikely that a large amount of energy could be injected or dissipated within a short time period, one



expects that the energy of successively visited unstable solutions are close to one another. At a coarse-grained level, it is then natural to assume that the energy  $\varepsilon$  evolves diffusively. In order to take into account the variation with  $\varepsilon$  of the number of unstable states, the evolution should also be biased toward values of the energy with a high entropy  $s(\varepsilon, R)$ . More precisely, the bias should depend on the derivative of the entropy with respect to the energy (a constant entropy introduces no bias in the dynamics). Altogether, the simplest evolution equation for the energy  $\varepsilon(t)$  incorporating the above ingredients is the following Langevin equation

$$\frac{d\varepsilon}{dt} = \gamma s'(\varepsilon, R) - \lambda + \xi(t) \quad (6)$$

where the prime denotes a derivative with respect to  $\varepsilon$ . To enforce the finite range of values  $\varepsilon_{\min} < \varepsilon < \varepsilon_{\max}$ , reflecting boundary conditions are assumed at  $\varepsilon_{\min}$  and  $\varepsilon_{\max}$ . The parameter  $\gamma$  is a proportionality coefficient to be determined later on, included for dimensional reasons. The term  $\lambda$  accounts for dissipative effects, and  $\xi(t)$  is a white noise describing the energy injection mechanism, satisfying

$$\langle \xi(t)\xi(t') \rangle = 2D \delta(t - t'), \quad (7)$$

where  $D$  is a diffusion coefficient in energy space. These are obviously strong simplifications: the dissipation rate could in principle depend on  $\varepsilon$  and the noise should rather be considered as colored and multiplicative in such non-equilibrium systems, but we wish to keep the model as simple as possible for the sake of illustration. The assumption of a constant dissipation rate is however justified in the limit where the width  $\varepsilon_{\max} - \varepsilon_{\min}$  of the accessible energy range is small with respect to  $\varepsilon_{\min}$ . Besides, considering that the noise is self-generated by the turbulent fluctuations, and thus results from the superposition of a number of contributions proportional to the volume  $V$ , one expects the diffusion coefficient to scale as  $D = D_0/V$ . Note that all parameters  $\gamma$ ,  $\lambda$  and  $D_0$  may depend on the Reynolds number  $R$ .

The Fokker-Planck equation describing the evolution of the probability distribution  $P(\varepsilon, R, t)$  then reads

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial \varepsilon} \left( [\gamma s' - \lambda] P \right) + \frac{D_0}{V} \frac{\partial^2 P}{\partial \varepsilon^2}. \quad (8)$$

The stationary solution  $P(\varepsilon, R)$  is obtained as

$$P(\varepsilon, R) = \frac{1}{Z} \exp \left( \frac{V}{D_0} (\gamma s(\varepsilon, R) - \lambda \varepsilon) \right). \quad (9)$$

Following standard statistical physics arguments, one expects the distribution  $P(\varepsilon, R)$  to be proportional to the number of unstable states  $\Omega_V(\varepsilon, R) \sim e^{Vs(\varepsilon, R)}$ , which imposes  $\gamma = D_0$ . Introducing the parameter  $\beta = \lambda/D_0$ , the stationary distribution then reads

$$P(\varepsilon, R) = \frac{1}{Z} \exp[V(s(\varepsilon, R) - \beta(R)\varepsilon)] \quad (10)$$

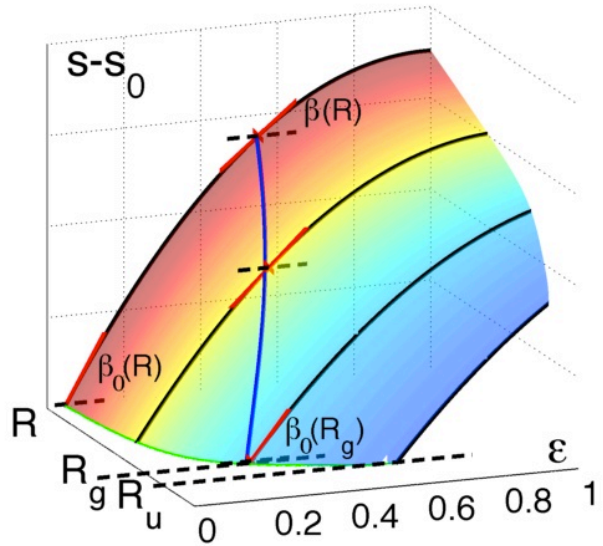


FIG. 3: (Color online) Sketch of the entropy surface and its slopes along the energy density direction, together with the path followed by the flow while varying the Reynolds number. Colors of the surface go from blue to red with increasing Reynolds number. The green line indicates  $\varepsilon_{\min}(R)$ . Fixing some Reynolds number  $R$ , a given dark line, one sets  $\beta(R)$ . Solving Eq. (11) then graphically amounts to finding a slope along the energy density direction equal to  $\beta(R)$ . A solution exists if  $\beta(R) < \beta_0(R)$ , the slope at the intersect with  $\varepsilon_{\min}(R)$ . Varying  $R$ , one follows the blue path on the surface, eventually leading to the value  $R_g$  such that  $\beta(R_g) = \beta_0(R_g)$ .

where we have emphasized the  $R$ -dependence of the parameter  $\beta$ , which in the present context describes the balance between the energy injection and the dissipation, as does the inverse temperature at equilibrium. If  $V$  is large, the distribution is dominated by the energy  $\bar{\varepsilon}(R)$  which maximizes the argument of the exponential, namely  $s(\varepsilon, R) - \beta(R)\varepsilon$ . If the maximum of  $s(\varepsilon, R) - \beta(R)\varepsilon$  lies within the interval  $\varepsilon_{\min}(R) < \varepsilon < \varepsilon_{\max}(R)$ , the most probable energy is the solution of

$$s'(\bar{\varepsilon}(R), R) - \beta(R) = 0. \quad (11)$$

Assuming the entropy  $s(\varepsilon, R)$  to be a concave function of  $\varepsilon$  (see figure 3),  $s'(\varepsilon, R)$  is a decreasing function of  $\varepsilon$ , and thus has its maximum at  $\varepsilon = \varepsilon_{\min}(R)$ .

We now introduce the key element of the model, which we borrow from the Random Energy Model (REM)[46, 47], one of the simplest disordered models exhibiting a freezing transition. The specificity of the REM, which is the ingredient leading to a glass transition in this model, is that the entropy has a finite slope at the minimum energy. We shall now see what are the consequences of this freezing (or glass) transition in the present model for turbulent flows.

By analogy, we thus assume that  $s'(\varepsilon, R)$  takes a finite value, denoted as  $\beta_0(R)$ , when  $\varepsilon \rightarrow \varepsilon_{\min}(R)$ . As a result, if  $\beta(R) < \beta_0(R)$ , Eq. (11) generally admits a solution  $\bar{\varepsilon}(R) > \varepsilon_{\min}(R)$ . In contrast, if  $\beta(R) > \beta_0(R)$ , Eq. (11) has no solution, and  $s(\varepsilon, R) - \beta(R)\varepsilon$  is maximum at  $\varepsilon = \varepsilon_{\min}(R)$ . The probability distribution then concentrates on  $\varepsilon_{\min}(R)$ . Intuitively, one expects  $\beta(R)$  to be a decreasing function of  $R$  (that is, the temperature  $\beta^{-1}$  characterizing the fluctuations increase with the Reynolds number). On the other hand, the total number of unstable states increases with the Reynolds number, and it is thus plausible that  $\beta_0(R)$  increases (or at least remains constant) with  $R$ . This suggests the existence of a Reynolds number  $R_g$  such that  $\beta(R_g) = \beta_0(R_g)$ . In this case, the average energy  $\bar{\varepsilon}(R)$  is larger than  $\varepsilon_{\min}(R)$  for  $R > R_g$ , while the dynamics in phase space concentrates on the states of minimal energy for  $R < R_g$ .

As emphasized at the beginning of this section, these conclusions hold under the unphysical hypothesis that no laminar state is present. However, if the paths leading from the unstable states to the laminar one are rare enough, the flow is likely to visit a large number of unstable states, and should thus partially equilibrate, before ending up into the laminar state. It is then plausible that the equilibrium distribution given in Eq. (10) qualitatively describes this quasi-equilibrium regime. A natural assumption is that most of the paths leading to the laminar state are connected to unstable states close to  $\varepsilon_{\min}(R)$ , the so-called edge states in the context of turbulence. As for  $R < R_g$ , the average energy remains close to  $\varepsilon_{\min}(R)$ , the flow should reach the laminar state in a reasonably short time. Conversely, for  $R > R_g$ , the typical energy remains well above the threshold  $\varepsilon_{\min}(R)$ , and one expects that it takes a very large time to find the laminar state, as it implies excursions very far from the typical energy.

Hence, the Reynolds number  $R_g$  appears as a transition (or crossover) value between a regime of short turbulent lifetime and a regime of large lifetime. Note also that the turbulent lifetime should essentially vanish below the Reynolds value  $R_u$  where unstable states cease to exist.

### C. Divergence of turbulent lifetime

In this section, we now try to put the above arguments on a more quantitative basis. We define the turbulent lifetime as the mean time to reach the laminar state after a sudden quench from a higher Reynolds number value, where turbulence is established. This situation can be modeled using Eq. (6) for the stochastic dynamics of  $\varepsilon(t)$ , with now an absorbing (instead of reflecting) boundary at  $\varepsilon = \varepsilon_{\min}$  to account for the presence of the laminar state. The initial condition at  $t = 0$  is chosen as  $\varepsilon(0) = \varepsilon_{\max}$ , to model the quench from high energy turbulent states. Determining the turbulent lifetime then amounts to computing the mean first passage time at the

absorbing boundary  $\varepsilon = \varepsilon_{\min}$ .

Such a calculation is however difficult for an arbitrary functional form of the entropy  $s(\varepsilon, R)$  and we have to restrict the choice of  $s(\varepsilon, R)$  to the linear form

$$s(\varepsilon, R) = \beta_0(R) (\varepsilon - \varepsilon_{\min}(R)) + s_0(R) \quad (12)$$

over the interval  $\varepsilon_{\min}(R) < \varepsilon < \varepsilon_{\max}(R)$ . In this case, the mean first passage time can be computed from the solution of the associated Fokker-Planck equation [75], and one finds

$$\tau = \frac{V}{D_0} (\Delta\varepsilon)^2 f(V(\beta_g - \beta) \Delta\varepsilon) \quad (13)$$

with  $\Delta\varepsilon = \varepsilon_{\max} - \varepsilon_{\min}$  and  $\beta_g = \beta(R_g)$ , and where the function  $f(x)$  is given by

$$f(x) = \frac{1}{x^2} (e^x - 1 - x). \quad (14)$$

For large  $V$ , the argument of the function  $f$  in Eq. (13) is large as soon as  $\beta_g \neq \beta$ , that is  $R \neq R_g$ . The value of  $\tau$  is then given, to a good approximation, by the asymptotic behavior of  $f(x)$  when  $x \rightarrow \pm\infty$ , which reads

$$f(x) \sim \frac{1}{|x|} \quad x \rightarrow -\infty, \quad (15)$$

$$f(x) \sim \frac{e^x}{x^2} \quad x \rightarrow +\infty. \quad (16)$$

Hence,  $\tau$  is given for  $\beta_g < \beta$  by

$$\tau \sim \frac{\Delta\varepsilon}{D_0(\beta - \beta_g)}, \quad (17)$$

which turns out to be independent of the volume  $V$ , as intuitively expected in the large  $V$  limit. In terms of Reynolds number, one thus has a power-law divergence for  $R$  close to  $R_g$  ( $R < R_g$ ),

$$\tau \sim \frac{\tau_0}{R_g - R}. \quad (18)$$

However, for any finite volume  $V$  this divergence is cut off when  $R$  approaches  $R_g$ , as soon as  $R_g - R \lesssim aV^{-1}$  with some constant  $a$ , and a crossover is observed to the exponential form obtained from Eq. (16)

$$\tau \sim \frac{e^{V(\beta_g - \beta)\Delta\varepsilon}}{D_0 V (\beta_g - \beta)^2}. \quad (19)$$

Contrary to Eq. (17), expression (19) involves the volume  $V$ . For  $V \rightarrow \infty$ ,  $\tau$  becomes infinite, and a true power-law divergence is observed for  $R < R_g$ . For very large but finite  $V$ , the divergence can be observed in practice only on a narrow range of Reynolds number, before  $\tau$  becomes exceedingly large. On this narrow range,  $(\beta_g - \beta)\Delta\varepsilon$  behaves linearly with  $R$ . In contrast, if  $V$  is not too large, the range of  $R$  over which the divergence is observed broadens, and corrections to the linear behavior

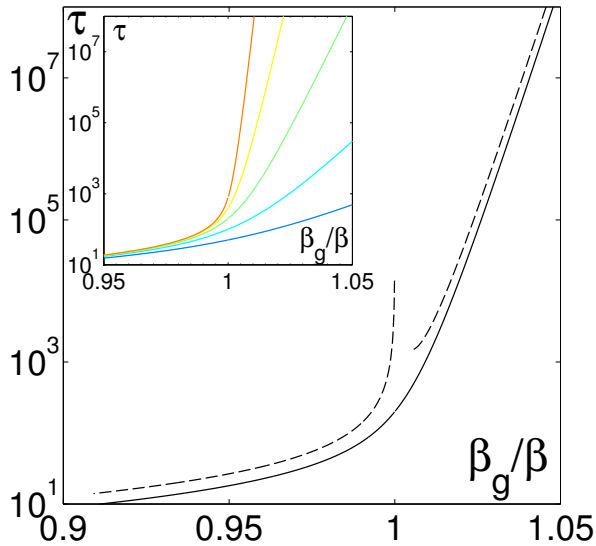


FIG. 4: (Color online) Sketch of the turbulent lifetime as a function of  $\beta_g/\beta$ , the effective Reynolds number. Main panel: the continuous curve is the lifetime for a given volume of the system as given by Eq. (13); dashed curves are the asymptotic functional forms which govern the behavior of  $\tau$  on each side of  $\beta_g$  -they have been shifted for clarity. Inset: Turbulent lifetimes for increasing system size: the larger the system, the steeper is the increase of lifetime. The singular behavior is observed in the limit of infinite system size only.

of  $(\beta_0 - \beta)\Delta\varepsilon$  with  $R$  can become observable, possibly leading to a super-exponential behavior of  $\tau$  as a function of  $R$ . Though sub-exponential behavior cannot be discarded, one expects at least  $\Delta\varepsilon$  to increase with  $R$ , which goes in favor of the super-exponential case.

#### IV. DISCUSSION

What did we learn from the physics of glasses?

The initial motivation of the analogy proposed in this paper was two-fold. First the intense debate that animated the transition to turbulence community regarding the possible divergence of the turbulent lifetimes at a finite Reynolds number was reminiscent of a similar situation encountered in the physics of glasses a few decades earlier. Second, the idea that the dynamics is controlled by unstable solutions away from the laminar state shared some similarity with the role played by the large number of saddles at the onset of the glass transition. The goal of the analogy presented here was to make these intuitions more precise.

We have shown that indeed, even with very good data, one cannot discriminate a singular dependence from a regular but very fast increase of the turbulent lifetimes, especially if one includes the possibility of a Vogel-

Fulcher-Tammann like singularity. We have also seen that finite size effects may lead to a crossover, which cannot be resolved experimentally or numerically because of the extremely large timescales at play. It is remarkable from that point of view that the model we introduced exhibits a crossover between the two main functional forms debated in the literature, namely the power-law and the super-exponential ones.

The model presented here was designed to be as simple as possible, and for the purpose of illustrating the analogy. As such, it does not pretend to be realistic in any way. Valuable improvements could be obtained by measuring in direct numerical simulations the dissipation rate as a function of the energy density, as well as characterizing the statistical properties of the turbulent energy fluctuations in the intermediate range of Reynolds numbers. In addition, counting the number of unstable solutions as a function of their energy density, that is accessing  $s(\varepsilon, R)$ , would be a major step towards the characterization of the transition to turbulence. This is obviously a difficult task, but still far less ambitious than characterizing the stability properties of these solutions and describing the complex interplay of their stable and unstable manifold. This simplification is in essence the one gained when switching from a dynamical systems point of view to a statistical physics one.

In the above section, we have considered  $V$  as the volume of the system. However, in the spirit of real space approaches the relevant volume to be considered may rather be the volume of coherent regions of the flow, namely regions over which correlations extend. In a very large aspect ratio experiment, it is plausible (though not obvious) that far away regions in the system experience no interactions. As a result, the volume  $V$  would acquire a more intrinsic nature: it would then be self-determined by the flow dynamics and not by the arbitrary size of the experiments.

Such a coherence volume cannot be accessed in the framework of models similar to the Random Energy Model, which is mean-field in nature. However, if the analogy with the physics of glasses proves to be fruitful, it would be of interest to consider its most recent developments (including in particular the Random First Order Transition scenario [76]), which precisely address the real-space description issue [43]. Pomeau [45] suggested more than twenty-five years ago that the growth and death of the laminar and turbulent regions could obey a first-order nucleation-like dynamics (albeit of a peculiar type, given the fluctuating active property of the turbulent state and the adsorbing character of the laminar state). Let us conclude with the somewhat naive suggestion that taking inspiration from the Random First Order Transition theory of glasses might be a way to extend the standard laminar-turbulent coexistence scenario to a situation where a large number of turbulent states (associated to local unstable solutions of the Navier-Stokes equation) coexist with the laminar state.



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